Approx Algo Examples

Note: All the examples are from chapter 35 of CLRS 3rd edition.

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Question 35-1:

a) I'll reduce partition to bin packing.

Given $(S, \frac{n}{2})$, an instance of partition, we want to construct in poly time (C, t), an instance of bin packing, s.t. partition is True iff bin packing is True.

Let $C_i = \dfrac{2S_i}{\sum_{j=1}^{m} S_j}$

If you add up $C_1, \dots, C_m$, you get 2.

Let $t = 2$ ← Hence, 2 bins are needed.

If there's a subset of S, S', whose sum is equal to $\frac{n}{2}$ then we can fit the objects into 2 bins.

Proof:

Let S' be a subset of S whose sum is $\frac{n}{2}$. Then, $S - S'$ also sums up to $\frac{n}{2}$.

Put the elements in S' into 1 bin and the elements of $S - S'$ into the other bin.

If we can distribute the objects in the 2 bins equally, then there's a subset of S that adds up to $\frac{n}{2}$.

Proof:

If we can distribute the objects into the 2 bins equally, then create S' and S'' from the objects in each bin respectively. Since the weights of each box is half the total weights, $S' = S'' = \frac{n}{2}$.

b) Suppose that the optimal number of bins is less than $\lceil S \rceil$.

We know that each S_i is between 0 and 1, exclusively, and each bin has a capacity of 1.

Without loss of generality, suppose that S is not a whole number. If it is, $\lfloor S \rfloor = \lceil S \rceil = S$.
Since S is not a whole number that means we need at least $\lfloor S \rfloor + 1$ bins to pack everything. However, this is equal to $\lceil S \rceil$, contradicting our earlier assumption.

c) Consider these 2 possibilities:
1. S is a whole number.
2. S is not a whole number.

For case 1, then all bins are completely filled, meaning that no bins are less than half full.

For case 2, suppose that 2 buckets have less than half capacity. Then, we can group the objects into 1 bucket, which means that the assumed soln is not optimal. Hence, at most 1 bin is less than half full.

Statement 1

d) We know in part c that at most 1 bin will be less than half full.

Let P be the number of bins used.

We know that $s > \frac{1}{2}(P-1)$ because of Statement 1.
$\frac{1}{2}(P-1)$ is the capacity.

$s > \frac{1}{2}(P-1)$

$2s > P-1$

$2s+1 > P$

If P were to be greater than $\lceil 2s \rceil$, that would cause a contradiction.

e) We know from part b that the opt number of bins required is at least $\lceil s \rceil$ and from part d that the num of bins used by the first-fit heuristic is never more than $\lceil 2s \rceil$.

$$
\begin{aligned}
P &\le \lceil 2s \rceil \\
&\le 2\lceil s \rceil \\
&\le 2P^*
\end{aligned}
$$

Question 35-3:

Our greedy strategy/algo is to pick the set that max the value of the num of uncovered points it covers divided by its weight.

Let C be the cover selected by the greedy algo.
Let C^* be the opt cover.

$$\left|\sum_{S_i \in C} w_i\right| = \sum_{x \in X} w_i c_x$$

$$\leq \sum_{S_i \in C^*} \sum_{x \in S_i} w_i c_x$$

$$\sum_{x \in S_i} w_i c_x = w_i \sum_{x \in S_i} c_x$$

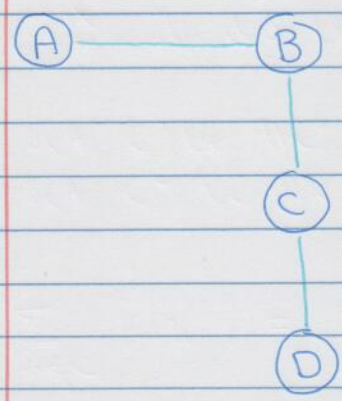
$$\leq w_i H(|S|)$$

$$\therefore \quad |C| \leq \sum_{S_i \in C^*} w_i H(|S_i|)$$

$$\leq \left|\sum_{S_i \in C^*} w_i\right| H(\max |S_i|)$$

Question 35-4:

a) Suppose we have the graph shown below.



(B, C) is a max matching while {(A, B), (C, D)} is a maximum matching.

b) Go through each edge and add it to the matching if it's compatible. Otherwise discard it. This takes $O(|E|)$ time.

c) The size of a maximum matching is a lower bound on the size of any vertex cover for G because in a maximum matching we get the edges s.t. no 2 edge can be connected to the same vertex. This means that the edges are disjoint. Furthermore, each vertex v in the VC covers at most 1 edge in the maximum matching.

$\therefore |M| \leq |VC|$

Maximum Matching

Vertex Cover

d) It consists of isolated vertices.
I.e. Vertices that are not connected to any other vertices.

e) If we take the endpoints of the edges in the maximum matching, we can create a vertex cover.

By part d, we know that every vertex must be included.
I.e. There are no vertices that isn't part of the vertex cover.

Furthermore, since the edges are disjoint, we know there are vertices.
↳ $2|M|$
$\therefore$ $2|M|$ is the size of a vertex cover in G.

f) We know in part c that the size of a maximum matching is a lower bound on the size of any vertex cover. Furthermore, we know in part e that $2|M|$ is the size of a vertex cover.

Let $|M|$ be the size of a maximal matching.
Let $|M^*|$ be the size of a maximum matching.
Let $|VC|$ be the size of a vertex cover.

$$|M^*| \le |VC|$$

$$2|M| = |VC|$$
$$\ge |M^*|$$

$$\frac{|M^*|}{|M|} \le 2$$